1st Dec. Calculas and Analytical Geometery

3rd Dec

Complex Numbers.

A number of the form $x+iy$ where $x,y \in R$. where x is real part and y is imaginary part.

eq. $2+3i \Rightarrow (2,3)$

MCQs

$\Rightarrow$ C.N does not hold order properties.

$\Rightarrow$ Every real number is a complex number with 0 as its imaginary part.

$15+0i$

Properties of complex Numbers.

Addition. $\Rightarrow (3,5)+(4,6)$

Multiplication $\Rightarrow (3,-1)\cdot(5,2)$

$(3-i)(5+2i) \Rightarrow 15+6i-5i-2i^2$

$= 17+i \Rightarrow (17,1)$

Division. $(3,2)\div(1,2)$.

$= \frac{3+2i}{1+2i} \times \frac{1-2i}{1-2i} \ldots\ldots$

Conjugate Complex Numbers.

If we have $x+iy$ then conjugate will be $x-iy$

$\Rightarrow$ Every Real Number is self conjugate.

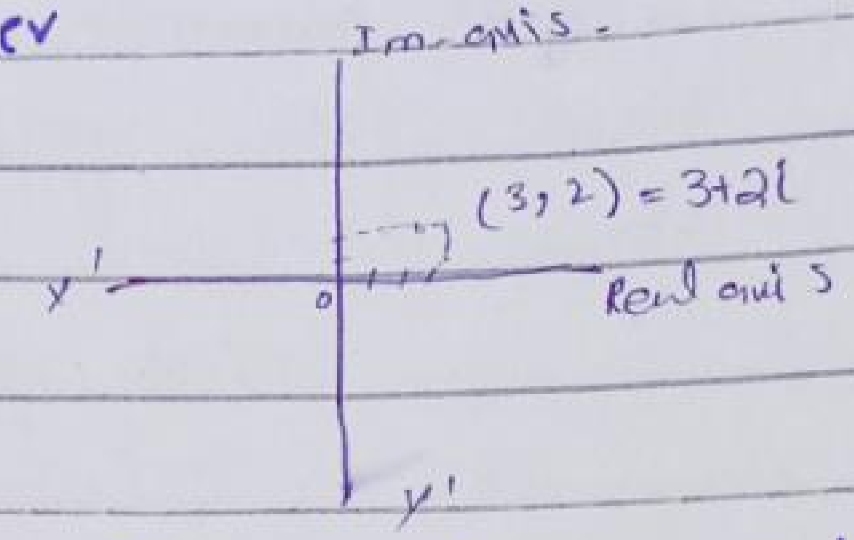
eg= $z=5$

$\bar{z}=5$

2nd lec

Calculus and analytical Geometry

Geometrical Interpretation of complex number



Modulus of complex number.

complex numbers a origion se distance modulus of complex number.

$$z = x + iy$$
$$|z| = \sqrt{x^2+y^2}$$

eg $z = 5 + 6i \Rightarrow |z| \sqrt{(5)^2+(6)^2}$

$= \sqrt{61}$

Polar form of complex number

$$x + yi = r\cos\theta + ir\sin\theta$$
$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1} y/x$$

eg $z = 1 + i\sqrt{3}$

$x = 1, \quad y = \sqrt{3}$

$1 + i\sqrt{3} = r\cos\theta + ir\sin\theta \rightarrow (1)$

$r = \sqrt{(1)^2+(\sqrt{3})^2}, \quad \theta = \tan^{-1}\frac{\sqrt{3}}{1}$

$\sqrt{4}, \quad \theta = 60^\circ$

$r = 2$, $\theta = 60°$

$1 + i\sqrt{3} = 2\cos 60° + i\,2\sin 60°$

De Moivers Theorem (Imp)

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \forall n \in \mathbb{Z}$

Application Simplify $(\sqrt{3} + i)^3$

firstly in polar form.

$\sqrt{3} + i = r\cos\theta + ir\sin\theta$

$x = \sqrt{3}$, $y = 1$.

$r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$

$\theta = \tan^{-1} y/x \Rightarrow \tan^{-1}\frac{1}{\sqrt{3}} \Rightarrow 30°$

$\sqrt{3} + i = 2\cos 30° + 2i\sin 30°$

Now applying De-Moivers theorm.

$= (2\cos 30° + 2i\sin 30°)^3$

$= (2(\cos 30° + i\sin 30°))^3$

$= 8(\cos 30° + i\sin 30)^3.$

$= 8(\cos 90° + i\sin 90°)$

$= 8(0 + i(1))$

$= 8i$

$= 0 + 8i$

$= (0, 8)$

$(1-\sqrt{3}i)^5$

$1-\sqrt{3}i = r\cos\theta + ri\sin\theta$

$x=1$, $y=-\sqrt{3}$

$r=\sqrt{(1)^2+(-\sqrt{3})^2}$, $\theta=\tan^{-1}\frac{-\sqrt{3}}{1}$

$r=\sqrt{1+3}$,

$r=\sqrt{4}$,

$r=2$, $r=-60^\circ$

$1-\sqrt{3}i = 2\cos(-60^\circ)+2i\sin(-60^\circ)$

Now apply De-moivers law.

$(1-\sqrt{3}i)^5 = (2\cos(-60^\circ)+i2\sin(-60^\circ))^5$

$= (2(\cos(-60^\circ)+i\sin(-60^\circ))^5$

$= 32(\cos(-60^\circ)+i\sin(-60^\circ))^5$

$60 \times 5 = 300$ $= 32(\cos(-300^\circ+i\sin(-300^\circ))$

$= 32\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$(1+\sqrt{3}i)^5$

$1+\sqrt{3}i = r\cos\theta + ir\sin\theta$

$x=1$, $y=\sqrt{3}$

$r=\sqrt{(1)^2+(\sqrt{3})^2}$, $\theta=\tan^{-1}\frac{\sqrt{3}}{1}$

$r=\sqrt{4}$, $\theta=\tan^{-1}\frac{1}{\sqrt{3}}$

$r=2$, $\theta=60^\circ$

$1+\sqrt{3}i = 2\cos 60^\circ + i2\sin 60^\circ$

Now apply De-moivers theorm

$= (2\cos 60^\circ + i2\sin 60^\circ)^5$

$= (2(\cos 60^\circ + i\sin 60^\circ))^5$

$= 32(\cos 300^\circ + i\sin 300^\circ)$

$= 32\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

Proof of De-moivers theoram.

Mathematical Induction

(i) Proof the given statement for $n=1$.

(ii) Assume it is true for $n=k$

(iii) Proof that it is true for $n=k+1$.

$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$.

$\cos(A-B) =$

$\sin(A-B) =$

Statement: if $n \in Z$ then.

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

proof for $n=1$.

L.H.S $= (\cos\theta + i\sin\theta)^1 \Rightarrow \cos\theta + i\sin\theta$.

R.H.S $= \cos 1\theta + i\sin 1\theta \Rightarrow \cos\theta + i\sin\theta$.

Hence it is true for $n=1$.

Siv waqar

Computer

Assume that it is true for $n=k$

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta \longrightarrow ①$$

Now for $n = k+1$.

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$$

L.H.S $= (\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)^1$

By equation ①

$= (\cos k\theta + i\sin k\theta) \cdot (\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$= \cos k\theta\cos\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta) - \sin k\theta\sin\theta$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k\theta + \theta) + i(\sin(k\theta + \theta))$

$= \cos(k+1)\theta + i(k+1)\theta$

$= R.H.S$

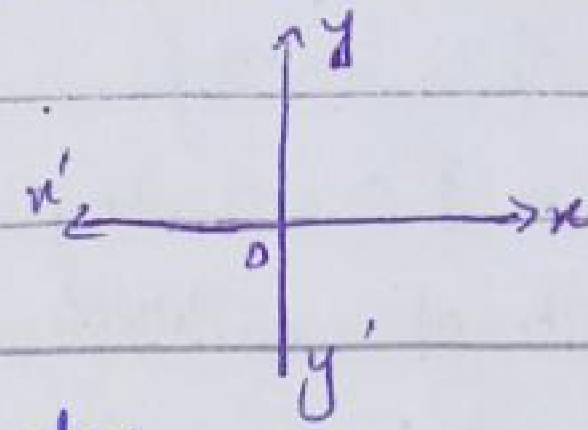
Assignment

simplify $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)^3$

By using De-Moveries theorem.

Simple Cartesian Curves

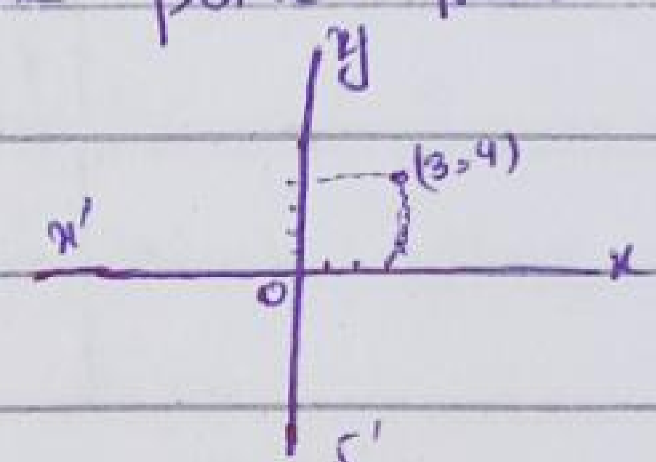
⇒ Cartesian plane



⇒ Cartesian Coordinates

Cartesian coordinates actually describe the distance of the point from origion ~~distance~~.

(3, 4)



- always draw in order pair.

⇒ Cartesian Product

$A = \{1, 2, 3\}$

$B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c),$
$(3, a), (3, b), (3, c)\}$

$B \times A = \ldots\ldots ?$

if only one statement is given to find cartesian product such as

$A = \{a, b, c\}$

then

$A \times A = \ldots$

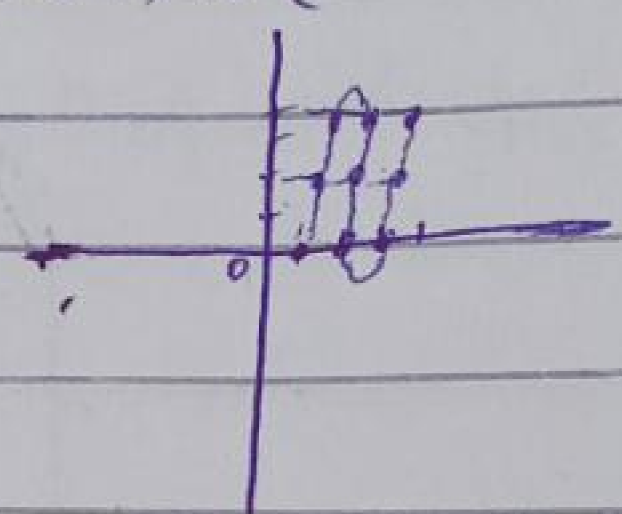
3rd lec

Graph of Cartesian Product

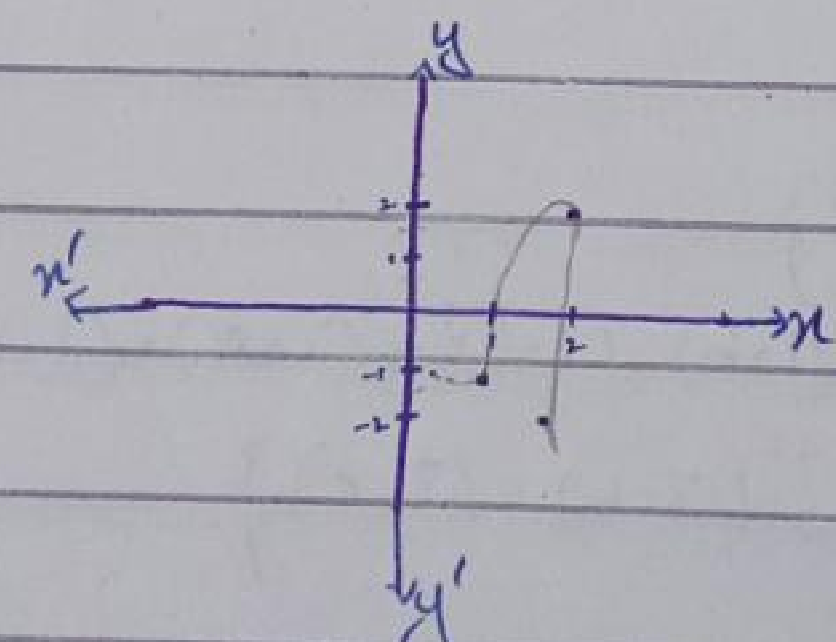
$A = \{1, 2, 3\}$
$B = \{0, 2, 4\}$

Graph of cartesian product A and B

$A \times B = \{(1,0), (1,2), (1,4), (2,0), (2,2),$
$(2,4), (3,0), (3,2), (3,4)\}$.



Show that Graph of cartesien plane $\{(+2, 2), (1,-1), (+2,-2)\}$ is a curve or straight line.



cartesian product of any sets can be represented by cartesian diagram we can plot the order pairs in by taking the first element along x-axis and second along y-axis in plane.
Each order pair is marked by point

Calculus and Analytical Geometry

Types of Simple Cartesian Curves.

1. Straight line

An equation of first degree in x and y is an equation of the form.

$$Ax + By + C = 0.$$

where A, B and C are constants.

eg $3x + 2y + 5 = 0$

Some special cases of straight line

(1) The slope intercept form.

$$y = mx + c$$

m is slop and c is y-intersept.

$$m = \tan\alpha \quad ,$$

$$m = \frac{y_2 - y'}{x_2 - x_1} \quad , \quad m = -\frac{a}{b}$$

(2) Two intercept form.

$$\frac{x}{a} + \frac{y}{b} = 1$$

(3) Normal form

$$x\cos\alpha + y\sin\alpha = P$$

(4) Point slope form.

$$y - y' = m(x - x_1)$$

(5) Two point form

$$\frac{y - y'}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

(6) Parametric form $\quad \frac{x - a}{\cos\alpha} = \frac{y - b}{\sin\alpha}$

(7) Eq of straight line passes through point of intersection of two lines.

$(ax_1+by_1+c_1)+k(ax_2+by_2+c_2)=0$

- Horizontal line

  parallel to x-axis

  (m) slop = 0

- Vertical line.

  Parallel to y-axis

  $m=\infty$

- if two line are parallel $\parallel$ (slope equal)

  $m_2=m_1$

- if two lines are perpendicular

  $m_1m_2=-1$.

Examples

l_1 $2x+y-4=0$

l_2 $x-5y-1=0$

l_3 $6x+8y-3=0$

l_4 $4x-3y-5=0$

(i) write down an equation of straight line

- parallel to l_1 and passing throog point(2,1)

slope of $l_1=-\frac{2}{1}=-2$

slope of req line $=-2$ ∴ lines are parallel

Equat

usi

(ii) wr

is

Equation of required line passing through $(2,1)$ and having slop

Using point slope form.

$$y-y_1=m(x-x_1)$$
$$y-1=-2(x-2).$$
$$y-1=-2x+4.$$
$$2x+y-5=0.$$

(ii) write down eq of straight line which is perpendicular to l_2 and passing through $(1,2)$

slop of $l_2 = \frac{\cancel{-}1}{\cancel{-}5} = \frac{1}{5}$

$m_1 m_2 = -1$. ∵ lines are perpendicular

$(\text{slope of } l_2)(\text{slope of req line}) = -1$.

$(\frac{1}{5})(\text{slope of req line}) = -1$.

Slop of req line $= -5$

Slope of Required line through $(1,2)$ and having slop -5

$$y-y_1=m(x-x_1)$$
$$y-2=-5(x-1)$$
$$y-2=-5x+5$$
$$5x+y-7=0.$$

write eq of straight line passing through Intersection of line and through $(2,3)$

l_1 and l_2

$3-14k=0$
$3=14k$

$(a x_1+b y_1+c_1)+k(a x_1+b y_1+c_2)=0$

$(2x+y-4)+k(x-5y-1)=0 \rightarrow (1)$

put $(2,3)$

$(2(2)+3-4)+k(2-5(3)-1)=0.$

$4+3-4+k(2-15-1)=0$

$3-14k=0.$

$14k=-3$

---

circle topic

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

I we

I do

Do I

Ayesh

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Does

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Calculus and Analytical Geometry

Function $y = f(x)$

Area of square $= l \times w$

$A = x \times x \Rightarrow x^2$

Area of circle $= \pi r^2$

In-put → machine → out-put

Domain — Range

set of real numbers

$F(t) = 2(t-1)+3$.

Find F at $0, 2, x+2$

$F(0) = 2(0-1)+3$

$= -2+3 \Rightarrow 1$.

Koi bhi function hai us ki domain Real numbers ho gi

$F(2) = 2(2-1)+3$

$= 5$

Always used open interval with infinity

Range Question pr depend kre gi

$F(x+2) = 2(x+2-1)+3$

$= 2x+2+3 \Rightarrow 2x+5$

Domain

$-\infty \leftarrow \cdots -3 \;\; -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \cdots \rightarrow +\infty$

Interval closed $[6, 10] = 6, 7, 8, 9, 10$.

open $(6, 10) = 7, 8, 9$

half open / half close $(6, 10] = 7, 8, 9, 10$.

half close / half open $[6, 10) = 6, 7, 8, 9$

$\{6, 10\}$ specific numbers

End

*

*

Example.

* $y = x$

Domain. $\mathbb{R}$, $(-\infty, \infty)$, R

Range. R

* $y = x^2$

Domain - $\mathbb{R}$

Range = $[0, \infty)$

* $y = \sqrt{x}$

Domain = $[0, \infty)$

Range = $[0, \infty)$

* $y = \sqrt{1-x^2}$

Domain = $[-1, 1]$

Range = $[0, 1]$

$f(t) = \frac{1}{\sqrt{t}}$

$f(t) = \sqrt{4-t^2}$

$\Rightarrow f(t) = \dfrac{1}{\sqrt{t}}$

Domain = $[1, \infty]$
Range = $[1, \infty]$

$\Rightarrow f(z) = \sqrt{4-(z)^2}$

Domain : $[-2, 2]$
Range = $[0, 2]$

Programming Fundamentals

Calculus and Analytical Geometry

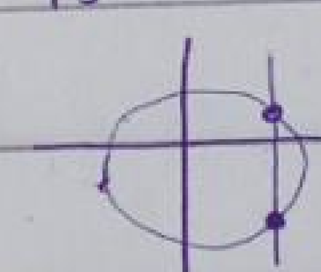
Graph of a function.

⇒ Graph of a function is acctually a graph of equation $y = f(x)$. and it consist of the points (x, y)

⇒ Not every curve represent a function

⇒ **vertical line test**

To analyse a curve we draw a vertical line if it intersects more then one point then it is not a graph of function.



Procedure

step 1 Make table of xy pair that specifies the function

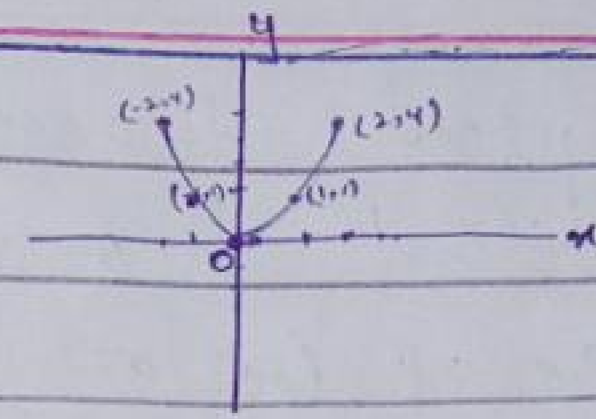
step 2 plot that pairs in cartsian plane.

step 3 Join these points to draw the graph

Examples:-

Graph the function $y = x^2$ over the interval $[-2, 2]$

x	$y=x^2$
2	4
1	1
0	0
-1	1
-2	4



Assignment $y=\sqrt{x}$ or $[-4, 4]$

$y = e^x$ $[-2, 2]$

Even and odd functions

Even, if $f(-x) = f(x)$

Odd, if $f(-x) = -f(x)$

Example:-

$f(x) = 3$

$f(-x) = 3$

neither Even nor odd

- $y = \frac{-1}{x^2}$

$f(x) = \frac{-1}{x^2}$

$f(-x) = \frac{-1}{(-x)^2}$

$= \frac{-1}{x^2}$ Even

- $f(x) = \frac{1}{|x|}$

Replace $x = -x$.

$f(-x) = \frac{1}{|-x|}$

$f(x) = \frac{1}{|x|}$

Even

made negative to positive bracket

- $f(x) = \frac{x}{x^2-1}$

$\frac{-x}{(-x)^2-1} = \frac{-x}{x^2-1}$

odd

* Piecewise defined function.

$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|2| = 2$

$|-3| = -(-3) = 3$

$$f(x) = \begin{cases} 3x+5 & \text{if } 0 \le x \le 5 \\ x^2-1 & \text{if } 6 \le x \le 9 \end{cases}$$

find value of $f(x)$ at $x=3$

$f(3) = 3(3)+5 = 14$

at $x = 8$

$(8)^2 - 1 = 63$

Limits of functions

$f(x) = \frac{1}{x-1}$

at $x = 1$

- Rules for finding the limits

⇒ Properties of limits :-

1) **Sum Rule**

$$\lim_{x\to c} [f(x) + g(x)] = L+M$$

eg $\lim_{x\to 3} (x^2+5x+3)$

$= (3)^2 + 5(3) + 9) = 27.$

2) Difference Rule

3) Product Rule

4) Constant Multiple Rule

$$\lim_{x\to 2} kf(x) = kL$$

$$= \lim_{x\to 2} 2(x^2-5)$$

$$= 2((2)^2 - 5) = -2$$

5) Quotient Rule

6) Power Rule $\lim_{x\to 5} (x-2)^5 \Rightarrow (5-2)^5 \Rightarrow 3^5$

Example

Exp

By

Exp

Example $f(x) = \sqrt{4x^2 - 3}$

at $x = -2$ find limit

$$= \lim_{x \to -2} \sqrt{4x^2 - 3}$$

Apply limit

$$= \sqrt{4(-2)^2 - 3} = \sqrt{16 - 3} = \sqrt{13}$$

Exp $\dfrac{x^2 + x - 2}{x^2 - 1}$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1}$$

$$\frac{(1)^2 + 1 - 2}{1^2 - 1} = \frac{0}{0} \text{ (Indetermined) (meaningless)}$$

- By eleminating its denominator algebrically

$$\text{limit}_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

$$\lim_{x \to 1} \frac{x^2 + 2x - x - 2}{x^2 - x} \Rightarrow \lim_{x \to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

Apply limit

$$= \frac{1+2}{1} = 3$$

Exp $\displaystyle \lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$

$\sqrt{2} - \sqrt{}$

$$\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \times \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}}$$

$$\lim_{h \to 0} \frac{(\sqrt{2+h})^2 - (\sqrt{2})^2}{h(\sqrt{2+h} + \sqrt{2})} \Rightarrow \lim_{h \to 0} \frac{\cancel{2} + \cancel{h} - \cancel{2}}{\cancel{h}(\sqrt{2+h} + \sqrt{2})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{\sqrt{2+0} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$(a+b)^3 = a^3+b^3+3ab(a+b)$

Exp $\lim_{v\to 2} \frac{v^3-8}{v^4-16} \Rightarrow \lim_{v\to 2} \frac{v^3-(2)^3}{(v^2)^2-(4)^2}$

$\lim_{v\to 2} \frac{(v-2)(v^2+2v+4)}{(v^2-4)(v^2+4)}$

$\lim_{v\to 2} \frac{\cancel{(v-2)}(v^2+2v+4)}{\cancel{(v-2)}(v+2)(v^2+4)}$

Apply limit

$\frac{4+2(2)+4}{(2+2)(4+4)} = \frac{12}{32} = \frac{3}{8}$

Exp

$\lim_{x\to 0} f(x) = 1$

$\lim_{x\to 0} g(x) = -5$

Evaluate $\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{2/3}}$

$= \frac{\lim_{x\to 0} [2f(x)-g(x)]}{\lim_{x\to 0} (f(x)+7)^{2/3}}$

= Apply limit

$\Rightarrow \frac{2(1)-(-5)}{(1+7)^{2/3}} \Rightarrow \frac{2+5}{(8)^{2/3}} \Rightarrow \frac{7}{4}$

Exp $\lim_{h\to 0} \frac{5}{\sqrt{5h+4}+2}$

By Rationalization

$\frac{5}{\sqrt{5(0)+4}+2}$

$\frac{5}{\sqrt{4}+4}$

$$\lim_{h\to 0} \frac{5}{\sqrt{5(h)+4}+2} \times \frac{\sqrt{5h+4}-2}{\sqrt{5h+4}-2}$$

$$\lim_{h\to 0} \frac{5(\sqrt{5h+4}-2)}{\sqrt{(5h+4)^2}-(2)^2} \Rightarrow \lim_{h\to 0} \frac{5\cancel{h}(\sqrt{5h}+4-2)}{5h+\cancel{4}-\cancel{4}}$$

$$\lim_{h\to 0} \frac{\sqrt{5(0)+4}-2}{0}$$

$$\lim_{h\to 0} \frac{0}{0}$$

Exp $$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$$

$\sqrt{(-1)^2+8}-3$

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \times \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$\lim_{x\to -1} \frac{\sqrt{(x^2+8)^2}-(3)^2}{(x+1)(\sqrt{x^2+8}+3)}$$

$$\lim_{x\to -1} \frac{x^2+8-9}{(x+1)\sqrt{x^2+8}+3} \Rightarrow \lim_{x\to -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$$

$$\lim_{x\to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(\sqrt{x^2+8}+3)} \Rightarrow \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3}$$

Apply limit

$$\frac{-1-1}{\sqrt{(-1)^2+8}+3} = \frac{-2}{\sqrt{9}+3} = \frac{-2}{3+3} = \frac{-2}{6}$$

Exp $$\lim \frac{x-1}{\sqrt{x+3}-2}$$

Calculus and Analytical Geometry

⇒ **Limit of a function:-**

Exp:- $\lim_{x \to 1} \left(\frac{1}{1-x} - \frac{3}{1-x^3} \right)$

$= \lim_{x \to 1} \left(\frac{1}{1-x} - \frac{3}{(1-x)(1+x+x^2)} \right)$

$= \lim_{x \to 1} \left(\frac{1+x+x^2-3}{(1-x)(1+x+x^2)} \right)$

$= \lim_{x \to 1} \left(\frac{x^2+x-2}{(1-x)(1+x+x^2)} \right)$

$= \lim_{x \to 1} \left(\frac{x^2+2x-x-2}{(1-x)(1+x+x^2} \right)$

$= \lim_{x \to 1} \left(\frac{(x-1)(x+2)}{(\cancel{1}-x)(1+x+x^2)} \right)$

$= \lim_{x \to 1} \left(\frac{-\cancel{(1-x)}(x+2)}{\cancel{(1-x)}(1+x+x^2)} \right)$

= Apply limit

$= -\frac{(1+2)}{1+1+(1)^2} \Rightarrow -\frac{\cancel{3}}{\cancel{3}} \Rightarrow -1$ Ans

$\because a^2-b^2 = (a+b)(a-b)$

Exp:- $\lim_{y \to x} \frac{y^{2/3} - x^{2/3}}{y-x}$

$= \lim_{y \to x} \left(\frac{(y^{1/3})^2 - (x^{1/3})^2}{y-x} \right)$

$= \lim_{y \to x} \frac{(y^{1/3} - x^{1/3})(y^{1/3} + x^{1/3})}{y-x}$

Geometry

$$= \lim_{y \to x} \frac{(y^{1/3} - x^{1/3})(y^{1/3} + x^{1/3})}{(y^{\frac{1}{3}})^3 - (x^{1/3})^3}$$

$\because a^3 - b^3 = (a+b)(a^2+ab+b^2)$

$$= \lim_{y \to x} \frac{(y^{1/3} - x^{1/3})(y^{1/3} + x^{1/3})}{(y^{1/3} - x^{1/3})(y^{2/3} + y^{1/3}x^{1/3} + x^{2/3})}$$

Apply limit

$$= \frac{x^{1/3} + x^{1/3}}{x^{2/3} + x^{1/3} \cdot x^{1/3} + x^{2/3}} \Rightarrow \frac{2x^{1/3}}{2x^{2/3} + x^{2/3}}$$

$\because x^{1/3} \cdot x^{1/3}$ Base same power add $x^{\frac{1}{3}+\frac{1}{3}} = x^{2/3}$

$$= \frac{2x^{1/3}}{3x^{2/3}} \Rightarrow \frac{2}{3} x^{1/3} \cdot x^{-2/3}$$

$$= \frac{2}{3} x^{-1/3} \Rightarrow \frac{2}{3x^{1/3}} \text{ Ans}$$

Exp:- $\lim_{x \to -1} \frac{x^{1/3} + 1}{x + 1}$

$$\lim_{x \to -1} \frac{x^{1/3} + 1}{(x^{1/3})^3 + (1)^3} \Rightarrow \lim_{x \to -1} \frac{\cancel{x^{1/3}+1}}{\cancel{(x^{1/3}+1)}(x^{2/3} - x^{1/3} + 1)}$$

$$= \lim_{x \to -1} \frac{1}{x^{2/3} - x^{1/3} + 1}$$

$$= \frac{1}{(-1)^{2/3} - (-1)^{1/3} + 1} \Rightarrow \frac{1}{((-1)^2)^{1/3} - (-1)^{1/3} + 1}$$

$$= \frac{1}{1 - (-1)^{1/3} + 1} = \frac{1}{2 - (-1)^{1/3}} \text{ Ans}$$

$x^2 + 4x - 2x - 8$

$x(x+4) - 2(x+4)$

Exp:- $\lim_{x \to -2} \frac{x^2 + 2x - 8}{x^2 - 4}$

$$\lim_{x \to -2} \frac{x^2 + 4x - 2x - 8}{(\ldots)(x-2)}$$

$$\lim_{x \to -2} \frac{x(x+4) - 2(x+4)}{(x-2)(x+2)}$$

$$\lim_{x \to -2} \frac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}(x+2)}$$

Apply limit

$$= \frac{-2+4}{-2+2} = \infty \text{ Ans}$$

Limits of piecewise functions:-

Exp

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x^3 & \text{if } x > 1 \end{cases}$$

Find limit at $x = 1$

$$L.H.L = \lim_{x \to 1^-} (x^2)$$

$$= (1)^2 \Rightarrow 1.$$

$$R.H.L = \lim_{x \to 1^+} (x^3)$$

$$1 \Rightarrow 1$$

$$L.H.S = R.H.S.$$

$$\lim_{x \to 1} f(x) = f$$

Exp

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ ax^2 & \text{if } x > -1 \end{cases}$$

Find limit at $x = -1$

$$L.H.S = \lim_{x \to 1^-} (x+2)$$

$$-1+2 = 1$$

$$R.H.S = \lim_{x \to -1^+} (ax^2)$$

$= a(-1)^2 = a$

Exp:- $f(x) = \begin{cases} 3 & \text{if } x \leq -2 \\ -\frac{1}{2}x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$

Find limit at $x = 2$ and $x = -2$

at $x = -2$.

L.H.L $\lim_{x \to -2^-} (3)$

$= 3$

R.H.L $\lim_{x \to -2^+} \left(-\frac{1}{2}x^2\right)$

$= -\frac{1}{2}(-2)^2$

$= -2.$

L.H.L $\neq$ R.H.L

So limit does not exist at $x = -2$.

At $x = 2$.

L.H.L $\lim_{x \to 2^-} \left(-\frac{1}{2}x^2\right)$

$= -\frac{1}{2}(2)^2$

$= -2.$

R.H.L $= \lim_{x \to 2^+} (3)$

$= 3$

L.H.L $\neq$ R.H.L

Limit does not exist at $x = 2$

Note

Absolute value function:-

$\Rightarrow |x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$

$\Rightarrow |x+3| = \begin{cases} x+3 & \text{if } x+3>0 \text{ or } x>-3 \\ -(x+3) & \text{if } x+3<0 \text{ or } x<-3 \end{cases}$

Exp:- $\lim_{x\to 3^-} \left(\frac{1}{x-3} - \frac{1}{|x-3|}\right)$

$= \lim_{x\to 3^-} \left(\frac{1}{x-3} - \frac{1}{-(x-3)}\right)$

$= \lim_{x\to 3^-} \left(\frac{1}{x-3} + \frac{1}{x-3}\right)$

$= \lim_{x\to 3^-} \left(\frac{1+1}{x-3}\right)$

$= \lim_{x\to 3^-} \left(\frac{2}{x-3}\right)$

Apply limit

$= \frac{2}{3-3} = \frac{2}{0} \Rightarrow \infty$ Ans

Exp:- $\text{limit}_{x\to 0} \frac{x}{x-|x|}$

$= \lim_{x\to 0^-} \frac{x}{x-(-x)}$

$\frac{x}{2x}$ $= \lim_{x\to 0^-} \frac{\cancel{x}}{2\cancel{x}}$

$= \frac{1}{2}$ Ans

(3) Programming

Exp:- $\lim_{h \to 1} \frac{|-1+h| - 1}{h}$

$\frac{-1+0-2}{h}$ -2

$\lim_{h \to 1} \frac{(-1+h) - 1}{h}$

$= \frac{(-1+1) - 1}{1} = -1$ Ans

Functional English

what is sentence

Sentence is a arrangement of words which means complete sense.

Structure of sentence

Every sentence has two parts

- Subject
- predicate.

answes always in integer and less then actuall number

Bracket function

$\Rightarrow x-1 \leq [x] \leq x$

$[1.5] = 1$

$[-3.5] = -4$

Limit of Bracket function

Exp $\lim_{x\to 0} [x][x+1]$

$-1 \quad 0 \quad 1$

L.H.L $= \lim_{x\to 0^-} [x][x+1]$

$= (-1)(-1+1)$

$= 0$

x approach to zero not actually equal to zero so accordig to bracket function we use -1

R.H.L $= \lim_{x\to 0^+} [x][x+1]$

$= (0)(0+1)$

L.H.L = R.H.L $= 0$

$\lim_{x\to 0} [x][x+1] = 0$

Exp $\lim_{x\to 1} [2x][x-1]$

L.H.L $\lim_{x\to 1^-} [2x](x-1)$

Apply limit

$= 2(0)(1-1)$

$= 0$

$0 \to 1 \quad 2$

we approch to 1 from left side so we take lower value 0.

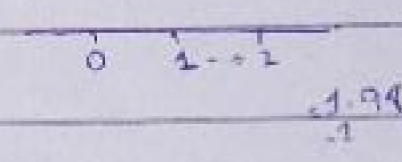
R.H.L $= \lim_{x\to 1^+} [2x](x-1)$

$0 \quad 1 \leftarrow 2$

$= 2(1)(1-1)$

$= 0$

Exp $\lim_{x\to 2} [x][x-5]$

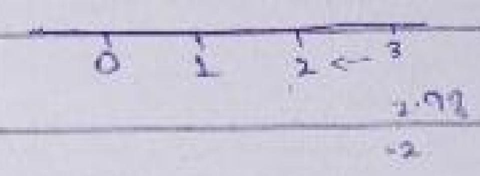


L.H.L $= \lim_{x\to 2^-} [x][x-5]$

$= (1)(1-5)$

$= -4$

R.H.L $= \lim_{x\to 2^+} [x][x-5]$



$= (2)(2-5)$

$= (2)(-3)$

$= -6$

L.H.L $\neq$ R.H.S

So limit does not exist.

Sandwich theorem:-

$f(x) \leq g(x) \leq h(x)$.

$\lim_{x\to a} f(x) = L$

$\lim_{x\to a} h(x) = L$

then by sandwich teorem

$\lim_{x\to a} g(x) = L$

Exp $\lim_{x\to 0} x\left[\frac{1}{x}\right]$

By definaction of Bracket function

$\frac{1}{x} - 1 \leq \left[\frac{1}{x}\right] \leq \frac{1}{x}$

Multiply by x

$x\left(\frac{1}{x} - 1\right) \leq x\left[\frac{1}{x}\right] \leq \frac{x}{x}$

$$= 1 - x < x\left(\frac{1}{x}\right) < 1$$

$$= \lim_{x \to 0} (1-x) \le \lim_{x \to 0} x\left(\frac{1}{x}\right) \le \lim_{x \to 0} 1$$

$$= 1 \le \lim_{x \to 0} x\left(\frac{1}{x}\right) \le 1.$$

By sandwich theorem.

$$\lim_{x \to 0} x\left(\frac{1}{x}\right) = 1.$$

Continuity

A function $f(x)$ is said to be continous at $x = a \in R$ if

$\Rightarrow f(x)$ is defined at $x = a$

$$\Rightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

$$\Rightarrow \lim_{x \to a} f(x) = f(a)$$

Exp

$$g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4-x^2 & \text{if } 1 \le x \le 10 \\ 6x^2+46 & \text{if } x > 10 \end{cases}$$

at $x = 1$, $x = 10$.

At $x = 1$

$$g(x) = -4 - x^2$$

$$g(1) = -4 - (1)^2$$

$$= -5$$

L.H.L $\lim_{x \to 1^-} (x^3)$

$$= 1$$

R.H.L $= \lim_{x \to 1^+} (-4 - x^2)$

apply limit

$= -4 - (+1)^2$

$= -5$

Since L.H.L $\neq$ R.H.L

So $g(x)$ is discountinous at $x = 1$

At $x = 10$

$g(x) = -4 - x^2$

$g(10) = -4 - (10)^2 \Rightarrow -104$

L.H.L $\operatorname{limit}_{x \to 10^-} (-4 - x^2)$

$= -4 - (10)^2 \Rightarrow -104$

R.H.L $\lim_{x \to 10^+} (6x^2 + 46)$

$6(10)^2 + 46 \Rightarrow 646$

$g(x)$ is discountinous at $x = 10$.

Exp $f(x) = \frac{x}{|x|}$ at $x = 0$

$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$

$\lim_{x \to 0^-} \frac{x}{-x} = -1$

$\lim_{x \to 0^+} \frac{x}{x} = 1$

This function is discountinous and undefined

Exp $g(x) = \begin{cases} -4 & \text{if } x \le 0 \\ |x-4| & \text{if } x > 0 \end{cases}$ $\quad |x-4| = \begin{cases} x-4 & \text{if } x > 4 \\ -x-4 & \text{if } x < 4 \end{cases}$

at $x = 0$

$g(x) = -4$

$g(0) = -4$

$L.H.L = \lim_{x \to 0^-} (-4) = -4$

$R.H.L = \lim_{x \to 0^+} |x-4|$

$|0-4| = 4$

This function is discontinous at $x = 0$

Exp $g(x) = \begin{cases} \sin x & x \le \pi/4 \\ \cos x & x > \pi/4 \end{cases}$

at $x = \frac{\pi}{4}$

$g(x) = \sin x$

$= \sin \frac{\pi}{4}$

$= 1/\sqrt{2}$

$L.H.L = \lim_{x \to \pi/4} (\sin x)$

$= \sin \frac{\pi}{4} = 1/\sqrt{2}$

$R.H.L = \lim_{x \to \pi/4^+} \cos x$

$\cos \frac{\pi}{4} = 1/\sqrt{2}$

This function is continous at $x = \pi/4$

Continuity

Exp $f(x) = \begin{cases} (1+3x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$

at $x = 0$.

$f(x) = e^2$

$f(0) = e^2$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+3x)^{1/x}$.

$= \lim_{x \to 0} \left[(1+3x)^{1/3x} \right]^3$

$= e^3$

Hence $f(x)$ is discontinus at $x = 0$

Exp $f(x) = \begin{cases} (1+2x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$

at $x = 0$.

$f(x) = e^2$

$f(0) = e^2$.

$= \lim_{x \to 0} (1+2x)^{1/x}$

$= \lim_{x \to 0} \left((1+2x)^{1/2x} \right)^2$

$= e^2$

Hence $f(x)$ is continus at $x = 0$

Exp $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

at $x = 0$ — limit [illegible] H.L [illegible] negative Infinity

$f(x) = 1$

$L.H.L = \lim_{x \to 0^-} \dfrac{e^{1/x}}{1+e^{1/x}}$ — Power [illegible] value [illegible] Infinity khud Infinity [illegible]

Apply Limit

$= \dfrac{e^{1/0}}{1+e^{1/0}} = \dfrac{e^{-\infty}}{1+e^{-\infty}}$

$= \dfrac{\frac{1}{e^{\infty}}}{1+\frac{1}{e^{\infty}}} \Rightarrow \dfrac{\frac{1}{\infty}}{1+\frac{1}{\infty}} \Rightarrow \dfrac{0}{1+0} = 0$

$R.H.L = \lim_{x \to 0^+} \dfrac{e^{1/x}}{1+e^{1/x}}$

$= \dfrac{e^{\infty}}{(1+e^{\infty})} \Rightarrow \dfrac{\cancel{e^{\infty}}}{\cancel{e^{\infty}}\left(\frac{1}{e^{\infty}}+1\right)}$

$= \dfrac{1}{\left(\frac{1}{\infty}+1\right)} = \dfrac{1}{0+1} = 1$

L.H.L $\neq$ R.H.L So $f(x)$ is discountinus at $x=0$.

$f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Exp $= \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

at $x = 0$

$f(x) = 1$

$f(0) = 1$

$= \lim_{x \to 0} \dfrac{\sin 2x}{x}$ $\quad\quad\quad \because \lim_{x \to 0} \dfrac{\sin x}{x} = 1$

$= \lim_{x \to 0} \dfrac{\sin 2x}{2x} \cdot 2$

$=$ when $x \to 0$

then

$2x \to 0$

$= 2(1) = 2$

Practice $f(x) = \begin{cases} \dfrac{\sin 2x}{\sin 2x} & \text{if } x \neq 0 \\ 2/3 & \text{if } x = 0. \end{cases}$

* $f(x) = \begin{cases} x + 4 & -6 \leq x \leq 2 \\ x & -2 \leq x < 2 \\ x - 4 & 2 < x < 6 \end{cases}$

Continuity:-

Points of Discontinuity

Exp

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ -4-x^2 & \text{if } 1 \le x \le 10 \\ 6x^2+46 & \text{if } x > 10 \end{cases}$$

At $x = 1$, $x = 10$.

At $x = 1$.

$f(x) = -4-x^2$

$f(1) = -4-(1)^2 = -5$

$L.H.L = \lim_{x \to 1^-} x^2$

$(1)^2 = 1$

$R.H.L = \lim_{x \to 1^+} -4-x^2$

$-4-(1)^2 = -5$

Hence this function is discontinuous at $x=1$

At $x = 10$.

$f(x) = -4-x^2$

$-4-(10)^2 = -104$.

$\lim_{x \to 10^-} (-4-x^2)$

$-4-(10)^2 = -104$.

$\lim_{x \to 10^+} (6x^2+46)$

$6(10)^2+46 = 646$.

Hence this function is discontinuous at $x=10$

Find the value of constant By the def of continuity.

Exe $f(x) = \begin{cases} x^3 & \text{if } x \le -1 \\ ax+b & \text{if } -1 \le x < 1 \\ x^2+2 & \text{if } x \geq 1 \end{cases}$

at $x = 1$

$f(x) = x^2 + 2$

$f(1) = (1)^2 + 2 = 3$

$L.H.L = \lim_{x \to 1} (ax+b)$

$= a+b$

$R.H.L = \lim_{x \to 1^+} (x^2+2)$

$= 3$

$\therefore f(x)$ is continous at $x = 1$

$a + b = 3 \rightarrow (1)$

At $x = -1$.

$f(x) = x^3$

$= -1$

$L.H.L = \lim_{x \to -1} (x^3)$

-1

$R.H.L = \lim_{x \to -1^+} (ax+b)$

$= a(-1) + b$

$= -a+b$

$\therefore f(x)$ is contin at $x = -1$

$-a+b=-1 \rightarrow (2)$

Add (1) and (2)

$$\begin{array}{r} \cancel{a}+b=3 \\ -\cancel{a}+b=-1 \\ \hline 2b=2 \end{array}$$

$b=1$

Put the value of b in (1).

$a+1=3$

$a=3-1=2$

For MCQ

Find the interval of continuity

Exp $f(x) = \dfrac{x^2-5}{x-1}$

∴ at $x=1$ $f(x)$ is undefine

So function is discountinous at $x=1$.

So interval of continuity will be $\mathbb{R}-\{1\}$

Exp $f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9} & \text{if } x \neq 3 \\ 6 & \text{if } x=3 \end{cases}$ at $x=3$

$f(x)=6$

$f(3)=6$

$\lim_{x \to 3} \dfrac{x^3-27}{x^2-9}$

$\lim_{x \to 3} \dfrac{\cancel{(x-3)} \cdot (x^2+3x+9)}{(x+3)\cancel{(x-3)}}$

sin and cos ki maximum range -1 or 1

Apply limit

$$= \frac{(3)^2 + 3(3) + 9}{2(3+3)}$$

$$= \frac{\cancel{27}}{\cancel{6}} = \frac{9}{2}$$

Exp $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

at $x = 0$

$f(x) = 0$

$f(0) = 0$

L.H.L = lim

Derivative

Derivative of a function w.r.t. 'x' is denoted by $f'(x)$ and can be defined as

$$\lim_{\delta x \to} \frac{f(x+\delta x) + f(x)}{\delta x} = f'(x)$$

Derivative of f at $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

L.H.D $\displaystyle \lim_{\delta x \to 0^-} \frac{f(x+\delta x) - f(x)}{\delta x}.$

R.H.D $\displaystyle \lim_{\delta x \to 0^+} \frac{f(x+\delta x) - f(x)}{\delta x}.$

Differentiable function

The function is said to be differentiable if its function exist.

Exp

Find the derivative of $f(x) = x^{2/3}$ at $x=0$

Sol: $f(x) = x^{2/3}$, $x = 0$

$f(0) = (0)^{2/3} = 0$

$$f'(0) = \lim_{x\to 0} \frac{x^{2/3} - 0}{x - 0}$$

$$= \lim_{x\to 0} \frac{x^{2/3}}{x}$$

$$= \lim_{x\to 0} x^{2/3 - 1}$$

$$= \lim_{x\to 0} x^{-1/3}$$

$$= \lim_{x\to 0} 1/x^{1/3}$$

Apply limit

$$= \frac{1}{(0)^{1/3}} = \frac{1}{0} \Rightarrow \infty$$

$\frac{2}{3}-1$

$\frac{2-3}{3}$

$-1/3$

Exp Find derivative of $f(x) = |x|$ at $x = 0$

$$f(x) = |x|$$

$$f(0) = |0| = 0$$

$$f'(0) = \lim_{x\to 0} \frac{|x| - 0}{x - 0}$$

$$|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$$

$$f'(0) = \lim_{x\to 0} \frac{|x|}{x} \longrightarrow (1)$$

$$\lim_{x\to 0^-} \frac{|x|}{x} \Rightarrow \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} \Rightarrow \lim_{x\to 0^+} \frac{x}{x} = 1$$

$$L.H.L \neq R.H.L$$

Hence ~~proved~~ $f'(0)$ does not exist at $x=0$.

Ex $y = \ln 3x$

Find $f'(x)$ by def.

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \qquad \because \lim_{x \to 0} (1+x)^{1/x} = e$$

$$\because \ln a - \ln b = \ln \frac{a}{b}$$

$$f(x) = \ln 3x$$

$$f(x+\delta x) = \ln 3(x+\delta x)$$

Now

$$f'(x) = \lim_{\delta x \to 0} \frac{\ln 3(x+\delta x) - \ln 3x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\ln \left(\frac{3(x+\delta x)}{3x} \right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\ln \left(\frac{x+\delta x}{x} \right)}{\delta x} \Rightarrow \lim_{\delta x \to 0} \frac{\left(\frac{x}{x} + \frac{\delta x}{x} \right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\ln \left(1 + \frac{\delta x}{x} \right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \ln \left(1 + \frac{\delta x}{x} \right) \qquad \because \ln a^b = b \ln a$$

Multiply and divide by x.

$$= \lim_{\delta x \to 0} \frac{x}{\delta x \cdot x} \ln \left(1 + \frac{\delta x}{x} \right)$$

$$= \lim_{\delta x \to 0} \frac{1}{x} \left[\frac{x}{\delta x} \ln \left(1 + \frac{\delta x}{x} \right) \right].$$

$$= \lim_{\delta x \to 0} \frac{1}{x} \ln \left(1 + \frac{\delta x}{x} \right)^{x/\delta x}$$

$$= \frac{1}{x} \ln \left[\lim_{\delta x \to 0} \left(1 + \frac{\delta x}{x} \right)^{x/\delta x}$$

$x > 0$

$: x < 0$

$= 0,$

$= \frac{1}{x} \ln e \qquad \because \ln e = 1$

$= \frac{1}{x}(1) = \frac{1}{x}$